New combinatorial principle on singular cardinals

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Abstract (Recall)

For an uncountable cardinal λ, we introduce a new combinatorial principle UB_λ to solve a problem about normal ideals over $\mathcal{P}_\kappa\lambda$.

The principle UB_λ is implied from a weak form of the square principle, however we see that UB_λ is consistent with almost all large cardinals and large cardinal properties. We also discuss other applications of UB_λ, for instance, $\mathrm{UB}_{\aleph_\omega}$ refutes $\langle \aleph_{\omega+1}, \aleph_\omega \rangle \twoheadrightarrow \langle \aleph_2, \aleph_1 \rangle$.

Motivation

λ always denotes an infinite cardinal.

Let μ be a cardinal. An ideal I over the infinite set A is weakly μ-saturated if there are no μ-many pairwise disjoint I-positive subsets.

Fact 1 (Folklore). *Let κ be a regular uncountable cardinal with $\kappa \leq \lambda$ and I a normal ideal over $\mathcal{P}_\kappa\lambda$.*

If $\mu \leq \lambda$ is a cardinal, then

$$I \text{ is weakly } \mu\text{-saturated} \iff I \text{ is } \mu\text{-saturated.}$$

If $\lambda^{<\kappa} = \lambda$ (e.g., GCH + $\mathrm{cf}(\lambda) \geq \kappa$) every ideal over $\mathcal{P}_\kappa\lambda$ is trivially weakly λ^+-saturated, but it might not be λ^+-saturated. Hence

$$\text{Weakly } \lambda^+\text{-saturated} \not\Longleftrightarrow \lambda^+\text{-saturated.}$$

Motivation

On the other hand, if $\mathrm{cf}(\lambda) < \kappa$ then every stationary subsets of $\mathcal{P}_\kappa\lambda$ has cardinality at least λ^+, and λ^+-many splitting is possible:

Fact 2 (Foreman–Magidor, Shioya). *Suppose* $\mathrm{cf}(\lambda) < \kappa < \lambda$. *Then* $\mathrm{NS}_{\kappa\lambda}$, *the non-stationary ideal over* $\mathcal{P}_\kappa\lambda$, *is not weakly* λ^+*-saturated.*

Fact 3 (U.). *Suppose* $\mathrm{cf}(\lambda) < \kappa$. *Then the existence of a weakly* λ^+*-saturated normal ideal over* $\mathcal{P}_\kappa\lambda$ *is a very strong property.*

Question

However, when $\mathrm{cf}(\lambda) < \kappa$, we do not know that

$$\text{Weakly } \lambda^+\text{-saturated} \iff \lambda^+\text{-saturated.}$$

Question 4. *Suppose* $\mathrm{cf}(\lambda) < \kappa < \lambda$. *Is every weakly* λ^+*-saturated normal ideal over* $\mathcal{P}_\kappa\lambda$ λ^+*-saturated?*

This question remains open. We will introduce a new combinatorial principle UB_λ and see that those saturation properties are equivalent under UB_λ.

The principle UB_λ

Definition 5. *Let* $S \subseteq \mathcal{P}(\lambda)$. $\mathsf{UB}_\lambda(S)$ *(UB stands for Un-Branched or Unique Branch or Usuba's Branching property,...) is the assertion that there exists* $f : {}^{<\omega}\lambda^+ \to \lambda^+$ *such that for every* $x, y \subseteq \lambda^+$*, if* x *and* y *are closed under* f*,* $x \cap \lambda = y \cap \lambda \in S$, $\mathsf{sup}(x) \leq \mathsf{sup}(y) \implies x \subseteq y$.

$\iff$ *For every large regular cardinal* θ*, a well-order* Δ *on* $H(\theta)$*, and* $M, N \prec \langle H(\theta), \in, \Delta, \lambda, S\rangle$*, if* $M \cap \lambda = N \cap \lambda \in S$ *and* $\mathsf{sup}(M \cap \lambda^+) \leq \mathsf{sup}(N \cap \lambda^+)$ *then* $M \cap \lambda^+$ *is an initial segment of* $N \cap \lambda^+$.

The principle UB_λ

Definition 5. *Let $S \subseteq \mathcal{P}(\lambda)$. $\mathsf{UB}_\lambda(S)$ (UB stands for Un-Branched or Unique Branch or Usuba's Branching property,...) is the assertion that there exists $f : {}^{<\omega}\lambda^+ \to \lambda^+$ such that for every $x, y \subseteq \lambda^+$, if x and y are closed under f, $x \cap \lambda = y \cap \lambda \in S$, $\mathrm{sup}(x) \leq \mathrm{sup}(y) \Longrightarrow x \subseteq y$.*

Note 6. ① *If $S = \{\lambda\}$, then $\mathsf{UB}_\lambda(S)$ holds; There is $f : {}^{<\omega}\lambda^+ \to \lambda^+$ such that for every f-closed $x \subseteq \lambda^+$, if $x \cap \lambda = \lambda$ then $x \in \lambda^+$.*

② *If $S \subseteq \mathcal{P}(\lambda)$ is non-stationary in $\mathcal{P}(\lambda)$, i.e., there exists $g : {}^{<\omega}\lambda \to \lambda$ such that there is no $x \in S$ which is closed under g, then $\mathsf{UB}_\lambda(S)$ holds in the trivial sense.*

Observations

Lemma 7. *For* $S \subseteq \mathcal{P}(\lambda)$, *if* S *is stationary in* $\mathcal{P}(\lambda)$, $\{\lambda\} \notin S$, *and* $|S| = \lambda$ *then* $\mathsf{UB}_\lambda(S)$ *fails.*

If $2^{<\lambda} = \lambda$ and $\mathrm{cf}(\lambda) > \omega$, then the set

$$S = \{x \subseteq \lambda : \sup(x) < \lambda\}$$

is stationary and has cardinality λ, hence $\mathsf{UB}_\lambda(S)$ fails. On the other hand, every stationary subsets of

$$\{x \subsetneq \lambda : \sup(x) = \lambda\}$$

has cardinality at least λ^+.

Definition 8. $\mathsf{UB}_\lambda \equiv \mathsf{UB}_\lambda(\{x \subseteq \lambda : \sup(x) = \lambda\})$.

UB_λ solves the problem

Lemma 9. *Let κ be a regular uncountable cardinal with* $\mathrm{cf}(\lambda) < \kappa < \lambda$. *Let I be a normal ideal over $\mathcal{P}_\kappa\lambda$. If I is weakly λ^+-saturated, then I is λ^+-saturated.*

Proof. We see only a special case that $I = \mathsf{NS}_{\kappa\lambda}|S$ for some stationary $S \subseteq \mathcal{P}_\kappa\lambda$. Notice that $\{x \in \mathcal{P}_\kappa\lambda : \sup(x) = \lambda\} \in I^*$.

Suppose that there is a family of stationary subsets $\mathcal{X} = \langle X_\xi : \xi < \lambda^+\rangle$ of S such that $X_\xi \cap X_\eta$ is non-stationary for $\xi \neq \eta$. We want to choose a family of clubs $\langle C_\xi : \xi < \lambda^+\rangle$ so that $(X_\xi \cap C_\xi) \cap (X_\eta \cap C_\eta) = \emptyset$.

Define $F : \mathcal{P}_\kappa\lambda \to \mathcal{P}(\lambda^+)$ as:

$$F(x) = \bigcup\{M \cap \lambda^+ : M \prec \langle H(\theta), \in, \Delta, \lambda, I, \mathcal{X}\rangle, M \cap \lambda = x\}$$

It is easy to see that for every $\xi < \lambda^+$, $\{x \in \mathcal{P}_\kappa\lambda : \xi \in F(x)\}$ contains a club.

Let $C_\xi = \{x \in \mathcal{P}_\kappa\lambda : \xi \in F(x)\} \in I^*$. Then $(X_\xi \cap C_\xi) \cap (X_\eta \cap C_\eta) = \emptyset$;

Suppose not and take $x \in (X_\xi \cap C_\xi) \cap (X_\eta \cap C_\eta)$. Then there are $M, N \prec \langle H(\theta), \ldots\rangle$ such that $M \cap \lambda = N \cap \lambda = x$, $\xi \in M$ and $\eta \in N$. If $\sup(M \cap \lambda^+) \leq \sup(N \cap \lambda^+)$, then $M \cap \lambda^+ \subseteq N \cap \lambda^+$ by UB_λ.

We have $\xi, \eta \in N$, thus there is a club $D \in N$ in $\mathcal{P}_\kappa\lambda$ with $X_\xi \cap X_\eta \cap D = \emptyset$. $x = N \cap \lambda \in D$ because D is club, hence $x \notin X_\xi \cap X_\eta$. This is a contradiction. $\square$

UB_λ is consistent with ZFC

Definition 10 (Cummings–Foreman–Magidor). ADS_λ *is the assertion that there is a family* $\{A_\xi : \xi < \lambda^+\}$ *such that*

① $A_\xi \subseteq \lambda$ *is unbounded in* λ *and* $|A_\xi| = \mathrm{cf}(\lambda)$.

② *For every* $\alpha < \lambda^+$, *there exists* $f : \alpha \to \lambda$ *such that* $\{A_\xi \setminus f(\xi) : \xi < \alpha\}$ *is a pairwise disjoint family.*

UB_λ is consistent with ZFC

Fact 11 (Shelah, Cummings–Foreman–Magidor). ① *If λ is regular, then ADS_λ holds.*

② *If λ is singular and $\square^*_\lambda$ holds, then ADS_λ holds.*
Hence it is consistent that ADS_λ holds for every λ.

③ *If λ is a singular cardinal with $\mathrm{pp}(\lambda) > \lambda^+$ (e.g., λ is a strong limit cardinal such that $\mathrm{cf}(\lambda) = \omega$ and $2^\lambda > \lambda^+$), then ADS_λ holds.*

④ *If κ is λ-supercompact cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$, then ADS_λ fails.*

⑤ *If Martin's Maximum holds, then ADS_λ fails for every λ with $\mathrm{cf}(\lambda) = \omega$.*

UB_λ is consistent with ZFC

Lemma 12. $\mathsf{ADS}_\lambda \Rightarrow \mathsf{UB}_\lambda$.

Proof. Choose $M, N \prec \langle H(\theta), \in, \Delta, \lambda \rangle$ such that $M \cap \lambda = N \cap \lambda$ and $\sup(M \cap \lambda) = \lambda$. We show that:

$$\sup(M \cap \lambda^+) \leq \sup(N \cap \lambda^+) \Rightarrow M \cap \lambda^+ \subseteq N \cap \lambda^+.$$

Take $\alpha \in M \cap \lambda^+$ and $\beta \in N \cap \lambda^+$ with $\alpha < \beta$. Let $\{A_\xi : \xi < \lambda^+\}$ be an ADS_λ-family which lies in $N \cap M$. Then there is $f \in N$ such that $f : \beta \to \lambda$ and $\{A_\xi \setminus f(\xi) : \xi < \beta\}$ is pairwise disjoint.

Since $A_\alpha \in M$ is unbounded in λ and $\sup(M \cap \lambda) = \lambda$, we know $A_\alpha \cap M$ is also unbounded in λ.

Fix $\gamma \in (A_\alpha \cap M) \setminus f(\alpha)$. $\gamma \in N$ since $M \cap \lambda = N \cap \lambda$.

Then α is definable in N; α is a unique ordinal $\alpha' < \beta$ satisfying $\gamma \in A_{\alpha'} \setminus f(\alpha')$. Hence $\alpha \in N$. $\square$

Good forcing notion adding UB_λ

Proposition 13. *Let* $\mathrm{cf}(\lambda) = \omega < \kappa < \lambda$ *and suppose that* κ *is* λ*-supercompact. Then there exists a poset* $\mathbb{P}$ *which satisfies the following:*

① $\mathbb{P}$ *is* σ*-directed closed and satisfies the* κ*-c.c.*

② $\mathbb{P}$ *forces "*$\kappa = \omega_2$ *and* UB_λ *holds".*

Outline of the proof

Notice that:

Lemma 14. *If* $\mathrm{cf}(\lambda) = \omega$*, the following are equivalent:*

① UB_λ.

② *There exists* $f : {}^{<\omega}\lambda^+ \to \lambda^+$ *such that for every* $x, y \in [\lambda^+]^\omega$*, if* x *and* y *are closed under* f*,* $x \cap \lambda = y \cap \lambda$ *and* $\sup(x) \leq \sup(y)$ *then* $x \subseteq y$.

Outline of the proof

Let $C = \{M \cap \lambda^+ : M \prec \langle H(\theta), \ldots \rangle\}$, $T = \{X \in C : \omega_1 \subseteq X, |X| < \kappa\}$.

Let $\mathbb{P}$ is the set of all pair $\langle f, p \rangle$ such that:

① $f : d(f) \times d(f) \to \omega_1$ for some $d(f) \in [\lambda^+]^\omega$.

② p is a function with $\mathrm{dom}(p) \in [T]^\omega$.

③ For every $X \in \mathrm{dom}(p)$,

❶ $p(X)$ is a $\subseteq$-increasing continuous sequence $\langle a_\xi : \xi \leq \alpha \rangle$ of $[d(f) \cap X]^\omega \cap C$ with length $\alpha < \omega_1$.

❷ For every $x \in [d(f) \cap X]^\omega \cap C$, if x is closed under f and $x \cap \lambda = a_\xi \cap \lambda$ for some $\xi \leq \alpha$ then $x \subseteq a_\xi$ (actually x is an initial segment of a_ξ).

$\mathbb{P}$ is σ-directed closed, satisfies the κ-c.c., and forces $\kappa = \omega_2$.

Outline of the proof

Let G be $(V, \mathbb{P})$-generic.

① $F = \{f : \exists p \, \langle f, p \rangle \in G\}$.

② For $X \in T$, $C_X = \bigcup \{p(X) : \exists f \, \langle f, p \rangle \in G, X \in \mathrm{dom}(p)$, x is F-closed$\}$.

Then

① $F : \lambda^+ \times \lambda^+ \to \omega_1$.

② C_X is a club in $[X]^\omega$ and for every $x \in [X]^\omega \cap C$, if x is closed under F, $x \cap \lambda = y \cap \lambda$ for some $y \in C_X$ then $x \subseteq y$.

Outline of the proof

Let $S = \{a \in [\lambda]^\omega$: there are $x_a, y_a \in C \cap [\lambda^+]^\omega$ such that x_a and y_a are closed under F, $x_a \cap \lambda = y_a \cap \lambda = a$, $\sup(x_a) \leq \sup(y_a)$ but $x_a \not\subseteq y_a\}$.

It is sufficient to show that S is non-stationary. Suppose to contrary that S is stationary. Since κ is λ-supercompact in V, a kind of stationary reflection principle of $[\lambda]^\omega$ holds;

There is $X \in T$ such that $S \cap [X \cap \lambda]^\omega$ is stationary in $[X \cap \lambda]^\omega$, and $a \in S \cap [X \cap \lambda]^\omega \Rightarrow x_a, y_a \subseteq X$.

Since S is stationary in $[X \cap \lambda]^\omega$ and C_X is a club in $[X]^\omega$, there is $a \in C_X$ such that $a \cap \lambda \in S$. Hence there are F-closed incomparable $x_a, y_a \in [X]^\omega \cap C$ such that $x_a \cap \lambda = y_a \cap \lambda = a$.. However this contradicts the choice of C_X.

Some conclusions

Lemma 15. *Let* $\mathrm{cf}(\lambda) = \omega < \kappa < \lambda$ *and suppose* κ *is* λ-*supercompact. Then* $\mathrm{UB}_\lambda(\{x \subseteq \lambda : x \cap \kappa \in \kappa\})$ *holds.*

Proof. By the previous proposition, there exists a poset $\mathbb{P}$ such that $\mathbb{P}$ satisfies the κ-c.c. and $\mathbb{P}$ forces UB_λ.

Let $\dot{f}$ be a name of a function witnessing UB_λ in the generic extension. By the κ-c.c. of $\mathbb{P}$, for each $s \in {}^{<\omega}\lambda^+$ there is $a_s \in [\lambda^+]^{<\kappa}$ such that $\Vdash \dot{f}(s) \subseteq a_s$. Then choose $g : {}^{<\omega}\lambda^+ \to \lambda^+$ so that for every g-closed $x \subseteq \lambda^+$ with $x \cap \kappa \in \kappa$, $\forall s \in {}^{<\omega}x\, (a_s \subseteq x)$.

It is easy to see that g witnesses $\mathrm{UB}_\lambda(\{x \subseteq \lambda : x \cap \kappa \in \kappa\})$. $\square$

Some conclusions

Corollary 16. *Let* $\mathrm{cf}(\lambda) < \kappa < \lambda$ *and suppose* κ *is* λ*-supercompact. Then* $\mathsf{UB}_\lambda(\{x \subseteq \lambda : x \cap \kappa \in \kappa\})$ *holds.*

Corollary 17. *Let* $\mathrm{cf}(\lambda) < \kappa < \lambda$ *and suppose* κ *is* λ*-supercompact. Let* $\mathrm{Col}(\omega, <\kappa)$ *be the standard poset which collapse* $\kappa = \omega_1$*. Then* UB_λ *holds in* $V^{\mathrm{Col}(\omega,<\kappa)}$.

Proof. Let $f : {}^{<\omega}\lambda^+ \to \lambda^+$ be a function witnessing $\mathsf{UB}_\lambda(\{x \subseteq \lambda : x \cap \kappa \in \kappa\})$. Then, because $\kappa = \omega_1$ in $V^{\mathrm{Col}(\omega,<\kappa)}$, it is easy to see that f witnesses UB_λ holds in $V^{\mathrm{Col}(\omega,<\kappa)}$. $\square$

Consistency of UB_λ with large cardinals

Corollary 18. *Let κ be supercompact. In $V^{\mathrm{Col}(\omega,<\kappa)}$,* UB_λ *holds for every singular cardinal λ with* $\mathrm{cf}(\lambda) = \omega$.

Proposition 19. *Relative to a certain large cardinal assumption, it is consistent that*

"ZFC + $\exists$supercompact cardinal + UB_λ *holds for every singular cardinal λ with* $\mathrm{cf}(\lambda) = \omega$*."*

Proof. Suppose there are two supercompact cardinals $\kappa_0 < \kappa_1$. In $V^{\mathrm{Col}(\omega,<\kappa_0)}$, UB_λ holds for every singular cardinal λ with $\mathrm{cf}(\lambda) = \omega$, and κ_1 remains a supercompact cardinal. $\square$

Consistency of UB_λ with large cardinals

This argument shows that UB_λ is consistent with *almost all* large cardinals; e.g.,

① λ is a limit of supercompact cardinals with $\mathrm{cf}(\lambda) = \omega$ + UB_λ holds.

② $\exists$superhuge cardinal + UB_λ holds for every λ with $\mathrm{cf}(\lambda) = \omega$,

③ There exists a non-trivial elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ and UB_λ holds, etc.

Consistency of UB_λ with large cardinals

Proposition 20. *Let κ be supercompact. Then in $V^{\mathsf{Col}(\omega_1,<\kappa)}$, UB_λ holds for every λ with $\mathrm{cf}(\lambda) = \omega_1$.*

Corollary 21. *Let $\kappa_0 < \kappa_1$ be supercompact. Then in $V^{\mathsf{Col}(\omega,<\kappa_0)\times\mathsf{Col}(\kappa_0,<\kappa_1)}$, UB_λ holds for every λ with $\mathrm{cf}(\lambda) \leq \omega_1$.*

Consistency of UB_λ with large cardinal properties

Proposition 22. *Relative to a certain large cardinal assumption, it is consistent that*

$$\text{“}ZFC + \textit{Martin's maximum} + \mathrm{UB}_{\aleph_\omega}.\text{”}$$

Proposition 23. *Relative to a certain large cardinal assumption, it is consistent that*

$$\text{“}ZFC + \langle \aleph_{\omega+1}, \aleph_\omega \rangle \twoheadrightarrow \langle \aleph_1, \aleph_0 \rangle + \mathrm{UB}_{\aleph_\omega}.\text{”}$$

Note 24. *The consistency of*

$$\langle\aleph_{\omega+1},\aleph_\omega\rangle \twoheadrightarrow \langle\aleph_1,\aleph_0\rangle$$

is known (Levinski–Magidor–Shelah), but the consistency of

$$\langle\aleph_{\omega+1},\aleph_\omega\rangle \twoheadrightarrow \langle\aleph_2,\aleph_1\rangle$$

is still open.

Other applications

Fact 25 (Folklore)**.** *Let κ be a regular uncountable cardinal with $\kappa \leq \lambda$. Let I be a normal ideal over $\mathcal{P}_\kappa\lambda$.*

① *If I is λ^+-saturated, then I is precipitous.*

② *If I is λ^+-preserving and $2^{\lambda^{<\kappa}} = \lambda^+$, then I is precipitous.*

Where an ideal I is μ-preserving if the standard generic ultrapower poset $\mathbb{P}_I$ associated with I forces that "μ remains a cardinal."

Other applications

Fact 25 (Folklore). *Let κ be a regular uncountable cardinal with $\kappa \le \lambda$. Let I be a normal ideal over $\mathcal{P}_\kappa\lambda$.*

① *If I is λ^+-saturated, then I is precipitous.*

② *If I is λ^+-preserving and $2^{\lambda^{<\kappa}} = \lambda^+$, then I is precipitous.*

Proposition 26 (UB_λ). *Suppose $\mathrm{cf}(\lambda) < \kappa < \lambda$. Let I be a normal ideal over $\mathcal{P}_\kappa\lambda$.*

① *If I is λ^{++}-saturated, then I is precipitous.*

② *If I is λ^{++}-preserving and $2^{\lambda^{<\kappa}} = \lambda^{++}$, then I is precipitous.*

Other applications

Fact 27 (Folklore)**.** *Suppose* cf$(\lambda) < \kappa < \lambda$*. Let* U *be a normal ultrafilter over* $\mathcal{P}_\kappa\lambda$*. If* M *is a ultrapower of* V *by* U*, then* $j``\lambda^+ \in M$*.*

Proposition 28 (UB$_\lambda$)**.** *Suppose* cf$(\lambda) < \kappa < \lambda$*. Let* I *be a normal precipitous ideal over* $\mathcal{P}_\kappa\lambda$*. If* M *is a generic ultrapower of* V *by a* $(V, \mathbb{P}_I)$*-generic filter, then* $j``\lambda^+ \in M$*.*

Other applications

Proposition 29 (UB$_\lambda$)**.** *Suppose* $\mathrm{cf}(\lambda) < \kappa < \lambda$*. Let* I *be a normal ideal over* $\mathcal{P}_\kappa\lambda$*. Then the following are equivalent:*

① I *is* λ^+*-saturated.*

② I *is weakly* λ^+*-saturated.*

③ *Every normal ideal* J *extending* I *is precipitous.*

④ *Every normal ideal* J *extending* I *is* λ^+*-preserving.*

Other applications

A cardinal λ is Jonsson if $\{x \subsetneq \lambda : |x| = \lambda\}$ is stationary in $\mathcal{P}(\lambda)$.

Fact 30 (Foreman)**.** *Suppose λ is Jonsson. Then there is no σ-complete λ^+-saturated ideal over $[\lambda]^\lambda$.*

Proposition 31. *Suppose λ is Jonsson. Then there is no weakly λ^+-saturated normal ideal over $[\lambda]^\lambda$.*

Other applications

Proposition 31. *Suppose λ is Jonsson. Then there is no weakly λ^+-saturated normal ideal over $[\lambda]^\lambda$.*

Proof. Suppose to contrary that there is a weakly λ^+-saturated normal ideal I over $[\lambda]^\lambda$.

If UB_λ holds, then I is in fact λ^+-saturated, this contradicts with Foreman's theorem.

Suppose UB_λ fails, then λ is a singular cardinal with $\mathrm{pp}(\lambda) = \lambda^+$. In this case, using Shelah's pcf-theory, we can derive a contradiction directly.

$\square$

Other applications

Proposition 32 (UB$_\lambda$). *Suppose λ is singular. Let S be the set $\{x \subseteq \lambda^+ : |x \cap \lambda| < |x|,\ |x \cap \lambda|$ is regular $> \mathrm{cf}(\lambda)$ and $\sup(x) = \lambda\}$. Then S is non-stationary in $\mathcal{P}(\lambda^+)$.*

Note 33. *For $n < \omega$,*
$\langle \aleph_{\omega+1}, \aleph_\omega \rangle \twoheadrightarrow \langle \aleph_{n+1}, \aleph_n \rangle$ holds $\iff$
$\{x \subseteq \aleph_{\omega+1} : \aleph_n = |x \cap \aleph_\omega| < |x|, \sup(x) = \aleph_\omega\}$ is stationary in $\mathcal{P}(\aleph_{\omega+1})$.

In particular,
UB$_{\aleph_\omega} \Rightarrow \langle \aleph_{\omega+1}, \aleph_\omega \rangle \not\twoheadrightarrow \langle \aleph_{n+2}, \aleph_{n+1} \rangle$ for every $n < \omega$.

Other applications

Proposition 32 (UB_λ). *Suppose λ is singular. Let S be the set $\{x \subseteq \lambda^+ : |x \cap \lambda| < |x|, |x \cap \lambda|$ is regular $> \mathrm{cf}(\lambda)$ and $\sup(x) = \lambda\}$. Then S is non-stationary in $\mathcal{P}(\lambda^+)$.*

Proof. Suppose to contrary that S is stationary. Then there is $M \prec \langle H(\theta), \ldots \rangle$ such that $|M \cap \lambda| < \left|M \cap \lambda^+\right|$, $|M \cap \lambda|$ is regular $> \mathrm{cf}(\lambda)$, and $\sup(M \cap \lambda) = \lambda$. Let $\mu = |M \cap \lambda|$.

Take a $\subseteq$-increasing continuous sequence $\langle a_\xi : \xi < \mu \rangle$ so that $\left|a_\xi\right| < \mu$, $\sup a_\xi = \lambda$, and $M \cap \lambda = \bigcup_{\xi < \mu} a_\xi$.

For $\xi < \mu$, let

$$A_\xi = \{\alpha \in M \cap \lambda^+ : \mathsf{SK}(a_\xi \cup \{\alpha\}) \cap \lambda = a_\xi\}.$$

Since μ is regular, $M \cap \lambda^+ = \bigcup_{\xi<\mu} A_\xi$.

$\left|M \cap \lambda^+\right| > \mu$, hence we can find $\xi^* < \mu$ such that $\left|A_{\xi^*}\right| > \mu$.

For $\alpha \in A_{\xi^*}$, let M_α the Skolem hull of $a_{\xi^*} \cup \{\alpha\}$ under $\langle H(\theta), \ldots\rangle$. By UB_λ, $\langle M_\alpha : \alpha \in A_{\xi^*}\rangle$ forms a chain with respect to $\subseteq$. Thus, $N = \bigcup_{\alpha \in A_{\xi^*}} M_\alpha$ is an elementary submodel of $\langle H(\theta), \ldots\rangle$. Then $N \cap \lambda = a_{\xi^*}$ and $\left|N \cap \lambda^+\right| \le |N \cap \lambda|^+ = |a_{\xi^*}|^+ \le \mu$. However $\left|N \cap \lambda^+\right| > \mu$ because $A_{\xi^*} \subseteq N \cap \lambda^+$, this is a contradiction. $\square$

Other applications

Proposition 33. *Let M, N be transitive models of ZFC with $M \subseteq N$. Let $\lambda \in M$ be such that*

$$M \models \text{“}\lambda \textit{ is a singular cardinal.”}$$

If $(\mathsf{UB}_\lambda)^M$ *holds and*

$$N \models \text{“}|\lambda|^N \textit{ is regular} > \mathrm{cf}^M(\lambda),\text{”}$$

then $(\lambda^+)^M \neq (\lambda^+)^N$.

Note 34. *In particular, if there are $M \subseteq N$ such that $\aleph^M_{\omega+1} = \aleph^N_2$, then* $\mathsf{UB}_{\aleph^M_\omega}$ *fails in M. However the existence of such models is unknown.*

Note 35. *If $\langle \aleph_{\omega+1}, \aleph_\omega \rangle \twoheadrightarrow \langle \aleph_2, \aleph_1 \rangle$ holds and Woodin cardinal exists, then there exist such models.*

Question

For a singular λ, is the failure of UB_λ consistent?

Note 36. *If* $\mathrm{cf}(\lambda) = \omega$, *then* UB_λ *is indestructible by forcing which preserves* λ *and* λ^+*;*

Lemma 37. *Suppose* $\mathrm{cf}(\lambda) = \omega$ *or* λ *is regular. If* UB_λ *holds then* $\Vdash_{\mathbb{P}}$ *"*UB_λ *holds" for every poset* $\mathbb{P}$ *which forces "*λ *and* $(\lambda^+)^V$ *are cardinals".*